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Paper - V

Topic - Laplacian, curl

This eqn represents ... curvilinear coordinates.

(3) Laplacian :- Laplacian may be obtained by combining eqn (15) and (27). Substituting

$$V = \nabla\psi(q_1, q_2, q_3) \text{ in eqn (27), we get}$$

$$\nabla \cdot \nabla\psi = \nabla^2\psi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left\{h_2 h_3 (\nabla\psi)_1\right\} + \frac{\partial}{\partial q_2}\left\{h_3 h_1 (\nabla\psi)_2\right\} + \frac{\partial}{\partial q_3}\left\{h_1 h_2 (\nabla\psi)_3\right\}\right]$$

Substituting components of $\nabla\psi$ from eqn (15), we get

$$\cancel{\nabla^2\psi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left\{\frac{h_2h_3}{h_1}\frac{(\nabla\psi)}{\partial q_1}\right\} + \frac{\partial}{\partial q_2}\left\{\frac{h_3h_1}{h_2}\frac{(\nabla\psi)}{\partial q_2}\right\} + \frac{\partial}{\partial q_3}\left\{\frac{h_1h_2}{h_3}\frac{(\nabla\psi)}{\partial q_3}\right\}\right]}$$

$$\nabla^2\psi = \left[\frac{1}{h_1h_2h_3}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\psi}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial q_3}\right)\right] \quad \text{—— (28)}$$

(4) Curl :- The curl of vector V is written as

$$\text{Curl } V = \nabla \times V = \nabla \times (\hat{u}_1 V_1 + \hat{u}_2 V_2 + \hat{u}_3 V_3)$$

$$= \nabla \times (\hat{u}_1 V_1) + \nabla \times (\hat{u}_2 V_2) + \nabla \times (\hat{u}_3 V_3) \quad \text{—— (29)}$$

Keeping in mind the relation curl (ϕA) $= \phi$ curl $A - A \times \text{grad } \phi$, we have

$$\text{curl } (\hat{u}_1 V_1) = \nabla \times (\hat{u}_1 V_1) =$$

$$= V_1 (\nabla \times \hat{u}_1) - \hat{u}_1 \times \nabla V_1$$

Substituting values of $\nabla \times \hat{u}_1$ and ∇V_1 from eqn (23a) and (15), we get

$$\nabla \times (\hat{u}_1 V_1) = V_1 \left[\frac{\hat{u}_2}{h_1 h_3} \frac{\partial h_1}{\partial q_2} - \frac{\hat{u}_3}{h_1 h_2} \frac{\partial h_1}{\partial q_2} \right]$$

$$- \hat{u}_1 \times \left[\frac{\hat{u}_1}{h_1} \frac{\partial V_1}{\partial q_1} + \frac{\hat{u}_2}{h_2} \frac{\partial V_1}{\partial q_2} + \frac{\hat{u}_3}{h_3} \frac{\partial V_1}{\partial q_3} \right]$$

$$= \frac{\hat{u}_2 V_1}{h_3 h_1} \frac{\partial h_1}{\partial q_3} - \frac{\hat{u}_3 . V_1}{h_1 h_2} \frac{\partial h_1}{\partial q_2} - \frac{\hat{u}_3}{h_2} \frac{\partial V_1}{\partial q_3} + \frac{\hat{u}_2}{h_3} \frac{\partial V_1}{\partial q_3}$$

$$= \hat{u}_2 \left(\frac{V_1}{h_3 h_1} \frac{\partial h_1}{\partial q_3} + \frac{1}{h_3} \frac{\partial V_1}{\partial q_3} \right) - \hat{u}_3 \left(\frac{V_1}{h_1 h_2} \frac{\partial h_1}{\partial q_2} + \frac{1}{h_3} \frac{\partial V_1}{\partial q_2} \right)$$

$$= \frac{\hat{u}_2}{h_3 h_1} \frac{\partial (V_1 h_1)}{\partial q_3} - \frac{\hat{u}_3}{h_1 h_2} \frac{\partial (V_1 h_1)}{\partial q_2} \quad \text{--- (30a)}$$

Either by similar treatment or by cyclic permutations of coordinates, we get

$$\nabla \times (\hat{u}_2 V_2) = \frac{\hat{u}_3}{h_1 h_2} \frac{\partial (V_2 h_2)}{\partial q_1} - \frac{\hat{u}_3}{h_2 h_3} \frac{\partial (V_2 h_2)}{\partial q_3}$$

$$\text{--- (30b)}$$

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$$\nabla \times (\hat{u}_3 V_3) = \frac{\hat{u}_1}{h_2 h_3} \frac{\partial (V_3 h_3)}{\partial q_2} - \frac{\hat{u}_2}{h_3 h_1} \frac{\partial (V_3 h_3)}{\partial q_1} \quad \text{--- (30c)}$$

Substituting these values in eqn (27), we get

$$\text{Curl } V = \nabla \times V = \frac{\hat{u}_1}{h_2 h_3}\left[\frac{\partial (V_3 h_3)}{\partial q_2} - \frac{\partial (V_2 h_2)}{\partial q_3}\right] + \frac{\hat{u}_2}{h_3 h_1}\left[\frac{\partial (V_1 h_1)}{\partial q_3} - \frac{\partial (V_3 h_3)}{\partial q_1}\right] + \frac{\hat{u}_3}{h_1 h_2}\left[\frac{\partial V_2 h_2}{\partial q_1} - \frac{\partial (V_1 h_1)}{\partial q_2}\right] \quad \text{--- (31a)}$$

In determinant form this may be written as

$$\text{Curl } V = \nabla \times V = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1 \hat{u}_1 & h_2 \hat{u}_2 & h_3 \hat{u}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ V_1 h_1 & V_2 h_2 & V_3 h_3 \end{vmatrix} \quad \text{--- (31b)}$$

Cor. In cartesian coordinates $ds^2 = dx^2 + dy^2 + dz^2$, therefore

$q_1 = x$, $q_2 = y$, $q_3 = z$ and $h_1 = h_2 = h_3 = 1$

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Also $\hat{u}_1 = i$, $\hat{u}_2 = j$ and $\hat{u}_3 = K$

Then $\text{grad}\, \psi = i \dfrac{\partial \psi}{\partial x} + j \dfrac{\partial \psi}{\partial y} + K \dfrac{\partial \psi}{\partial z}$

$$\text{div}\, V = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

and $\text{curl}\, V = \left(\dfrac{\partial V_z}{\partial y} - \dfrac{\partial V_y}{\partial z}\right) i + \left(\dfrac{\partial V_x}{\partial z} - \dfrac{\partial V_z}{\partial x}\right) j$

$$+ \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right) K$$